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An elliptic superpotential for softly broken $\mathcal{N} = 4$ supersymmetric Yang-Mills theory

Nick Dorey*

Department of Physics, University of Washington, Box 351560,
Seattle, Washington 98195-1560, USA
Email: dorey@fermi.phys.washington.edu

ABSTRACT: An exact superpotential is derived for the $\mathcal{N} = 1$ theories which arise as massive deformations of $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory. The superpotential of the SU(N) theory formulated on $\mathbb{R}^3 \times S^1$ is shown to coincide with the complexified potential of the N-body elliptic Calogero-Moser hamiltonian. This superpotential reproduces the vacuum structure predicted by Donagi and Witten for the corresponding four-dimensional theory and also transforms covariantly under the S-duality group of $\mathcal{N} = 4$ SYM. The analysis yields exact formulae with interesting modular properties for the condensates of gauge-invariant chiral operators in the four-dimensional theory.

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*Department of Physics, University of Wales Swansea, Singleton Park, Swansea, SA2 8PP, UK.

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1. Introduction and review

Four-dimensional gauge theories with $\mathcal{N}=4$ supersymmetry are believed to provide an exact realization of the duality between electric and magnetic charges originally proposed by Montonen and Olive [1]. The extent to which this duality is preserved in theories with less supersymmetry, obtained as massive deformations of the $\mathcal{N}=4$ theory, is an important and interesting question. For deformations which preserve $\mathcal{N}=2$ supersymmetry, the exact solution of the corresponding low-energy theory is determined by a complex curve which is manifestly invariant under $\mathrm{SL}(2,\mathbb{Z})$ transformations. The relevant curve was given by Seiberg and Witten for gauge group SU(2) [2], and subsequently generalized to the SU(N) case by Donagi and Witten [3]. The singularities of these curves also determine the vacuum structure of the theories obtained by breaking $\mathcal{N}=4$ supersymmetry down to an $\mathcal{N}=1$ subalgebra. In terms of $\mathcal{N}=1$ supersymmetry, these theories contain a vector multiplet and three massive adjoint chiral multiplets. Donagi and Witten used this correspondence to show that the $\mathcal{N}=1$ theory with gauge group SU(N) has a total of $\sum_{d|N} d$ massive vacua which are permuted by S-duality transformations. For $N>2$, the theory also has massless vacua.

In this paper, I will present an alternative analysis of the $\mathcal{N}=1$ theories described above. The approach followed, based on that of [5], is to compactify the four-dimensional theory on a circle of radius R. As we review below, the SU(N) theory then has a Coulomb branch parametrized by N chiral superfields,[1] X^a, which take values on a torus whose complex structure parameter τ is the complexified gauge coupling. The main result is that the exact superpotential of the theory is

$$\mathcal{W}=m_1m_2m_3\sum_{a>b}\mathcal{P}(X_a-X_b)\,, \tag{1.1}$$

where m_i, $i=1,2,3$ are the masses of the adjoint chiral multiplets and $\mathcal{P}$ is the Weierstrass function [6]. Note that $\mathcal{W}$ does not depend on R. In the limit $R\to\infty$, comparison with the four-dimensional results of [3] provides a detailed test of this proposal. In particular, we will see that $\mathcal{W}$ has a set of critical points which match the singular points of the Donagi-Witten curves. The superpotential (1.1) transforms with weight two under SL$(2,\mathbb{Z})$ modular transformations and the action of S-duality on the massive vacua predicted in [3] is manifest in this approach.

For massive theories with $\mathcal{N}=1$ supersymmetry we typically expect to obtain exact results for the condensates of gauge-invariant chiral operators [7, 8]. Where a description in terms of soft breaking of $\mathcal{N}=2$ supersymmetry exists, such results can be obtained directly from the corresponding complex curve. For example consider the condensates $u_n=\langle \mathrm{Tr}\Phi^n\rangle$, $n=2,\ldots,N$, where Φ is one of the adjoint scalar fields of the SU(N) theory. In principle, the values of the u_n in each vacuum be infered from the location of the corresponding singular point in the moduli space of the Donagi-Witten curve. In practice, this is approach is not straightforward because there is no explicit formula for the SU(N) curve. As a simple application of the superpotential described above, I calculate the exact value of the condensate u_2 in each massive vacuum of the SU(N) theory when N is prime. The resulting set of vacuum values transforms with weight two under SL$(2,\mathbb{Z})$, modulo the expected permutations of the vacua. For a related approach to calculating the gluino condensate in $\mathcal{N}=1$ supersymmetric Yang-Mills theory see [9]. Additional recent discussion of the superpotential of $\mathcal{N}=1$ SYM in four dimensions is given in [10]. Another interesting aspect of these results is that the superpotential (1.1) coincides with the (complexified) potential of the elliptic Calogero-Moser hamiltonian. The latter is the integrable system associated with the Donagi-Witten solution of the four-dimensional $\mathcal{N}=2$ theory mentioned above [11]. This reveals an interesting connection between $\mathcal{N}=1$ theories and integrability which is sketched in the final section of the paper.

[1] Some conventions used throughout this paper: Cartan algebra valued fields for SU(N) such as $X=X^aH^a$ have N components X^a, $a=1,\ldots,N$, which are subject to the constraint $\sum_{a=1}^{N}X^a=0$. The only exception is for gauge group SU(2) where we work with a single unconstrained field $X=X^2-X^1$. Supersymmetries are counted with four-dimensional conventions throughout. Thus $\mathcal{N}=1$ always denotes a theory with four supercharges.

1.1 The four-dimensional theory

We begin by considering $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) theory with gauge group SU(N) in four dimensions. In terms of $\mathcal{N}=1$ superfields this theory contains a gauge multiplet V as well as three chiral multiplets Φ_i, $i=1,2,3$ in the adjoint representation of SU(N). The superspace lagrangean includes a superpotential $W=\mathrm{Tr}(\Phi_1[\Phi_2,\Phi_3])$ which preserves an SU(4) R-symmetry but explicitly breaks the potentially anomalous global U(1) under which the Φ_i each transform with charge +1. Soft breaking to $\mathcal{N}=1$ is accomplished by introducing masses m_i for each chiral superfield Φ_i. Including these terms the superpotential becomes

$$W=\mathrm{Tr}\left(\Phi_1[\Phi_2,\Phi_3]+m_1\Phi_1^2+m_2\Phi_2^2+m_3\Phi_3^2\right). \tag{1.2}$$

For generic values of the masses m_i, the SU(4) R-symmetry is completely broken. It is convenient to rescale the chiral superfields as $\Phi_1=2\sqrt{m_2m_3}X$, $\Phi_2=2\sqrt{m_3m_1}Y$, $\Phi_3=-2\sqrt{m_1m_2}Z$. Up to an overall factor, the superpotential becomes

$$W=\mathrm{Tr}\left(\frac{1}{2}\left(X^2+Y^2+Z^2\right)-X[Y,Z]\right). \tag{1.3}$$

Following [12, 3], we will now determine vacuum structure of the classical theory with superpotential (1.2). To do this we must impose the F-term equations which come from varying W with respect to X, Y and Z. The first equation is $X=[Y,Z]$ and the other two are generated by cyclic permutation of the three superfields. A supersymmetric vacuum is therefore described by three $N\times N$ matrices which obey the commutation relations of an SU(2) Lie algebra. However, we still have to impose the corresponding D-term equations and mod out by SU(N) gauge transformations. As usual these two steps can be combined by dividing out the action of the complexified gauge group SL($N,\mathbb{C}$) on X, Y and Z. In fact, up to an SL($N,\mathbb{C}$) gauge transformation, there is exactly one solution of the SU(2) commutation relations in terms of $N\times N$ matrices for each N-dimensional representation of SU(2). Except for the unique irreducible representation of dimension N, each representation can be decomposed as the sum of irreducible pieces of dimensions $n_1,n_2,\ldots,n_r$ with $\sum_l n_l=N$. In general, unless the representation is trivial, the corresponding non-zero values of the adjoint scalar fields break the gauge group SU(N) down to a subgroup H. The vacuum structure for each representation ρ is essentially determined by H and may be analysed as follows:

(1) If ρ is the irreducible representation, the gauge group is completely broken. In this case the theory is in a Higgs phase with a mass gap. As the theory remains weakly coupled at all energy scales, the semiclassical analysis can be taken at face value and we find exactly one vacuum.

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(2) If ρ is trivial, the gauge group is completely unbroken and the theory is in a confining phase. At low energies, the adjoint matter multiplets decouple and the theory flows to $\mathcal{N} = 1$ SYM. The effective theory therefore becomes strongly-coupled in the IR and generates mass gap. In particular $\mathcal{N} = 1$ SYM has an Z_{2N} global symmetry which is spontaneously broken to Z_2 by gluino condensation giving N supersymmetric vacua [7, 8]. Cases where ρ is the sum of d irreducible representations each of the same dimension also lead to confinement. In these cases $H = \mathrm{SU}(d)$ and the theory flows to $\mathcal{N} = 1$ SYM with this gauge group. By similar reasoning this gives d massive supersymmetric vacua.

(3) In the remaining cases ρ includes irreducible pieces of different dimensions and the unbroken gauge group includes abelian factors. This means that the theory is realized in a Coulomb phase with at least one massless photon.

Combining case (1) and (2) we find that the total number of massive vacua of the $\mathrm{SU}(N)$ theory is the sum of the divisors of N. The case where N is prime is particularly straightforward as only the irreducible and the trivial representations contribute and hence there are exactly $N + 1$ massive vacua. In particular, the above analysis reveals that the $\mathrm{SU}(2)$ theory has three massive vacua, one in the Higgs phase and two in the confinement phase. The possibility of massless vacua first arises for gauge group $\mathrm{SU}(3)$ where ρ can be chosen as a sum of one copy of the trivial representation and one of the fundamental. In this configuration the unbroken gauge group is $\mathrm{U}(1)\times\mathrm{SU}(2)$. Donagi and Witten showed that choice leads to a single supersymmetric vacuum in the Coulomb phase.

1.2 Compactification to three dimensions

We now consider what happens if one compactifies the four-dimensional theory introduced above to three dimensions on a circle of radius R. The most obvious modification is that the theory acquires new scalar degrees of freedom which come from the Wilson lines of the four-dimensional gauge field around the compact dimension. Non-zero values for these scalars generically break the gauge group down to its maximal abelian subalgebra; $\mathrm{SU}(N) \to \mathrm{U}(1)^{N-1}$. Hence, at the classical level, the compactified theory has a Coulomb branch. By a gauge rotation, the Wilson line can be chosen to lie in the Cartan subalgebra, $\phi = \phi^a H^a$. Each scalar ϕ^a is related to the corresponding Cartan component of the four-dimensional gauge field as $\phi^a = \int_{S^1} A^a \cdot dx$. As a consequence, ϕ^a can be shifted by an integer multiple of 2π by performing a topologically non-trivial gauge transformation which is single-valued in $\mathrm{SU}(N)/Z_N$. As there are no fields in the theory which are charged under the center of the gauge group, it is consistent to divide out by all such gauge transformations. Thus we learn that each ϕ^a is a periodic variable with period 2π.

In addition to the N periodic scalars ϕ^a, the low energy theory on the Coulomb branch also includes N massless photons A^a_i, with $\sum_{a=1}^{N} A^a_i = 0$, where $i = 1, 2, 3$

is a three-dimensional Lorentz index. In three dimensions, a massless abelian gauge field can be eliminated in favour of a scalar by a duality transformation. Thus we can eliminate the gauge fields A_i^a, in favour of N scalars σ^a subject to $\sum_{i=1}^{N} \sigma^a = 0$. In particular, the fields σ^a appear in the effective action as lagrangean multipliers for the Bianchi identity: $\mathcal{L}_\sigma \sim \sigma\varepsilon^{ijk}\partial_i F_{jk}$. After integrating by parts, we obtain a term which only depends on $\partial_i\sigma^a$ not on σ^a itself. However, integrating by parts also gives rise to a surface term in the action which is proportional to the magnetic charge. Specifically, the normalization of σ^a is chosen so that the resulting term in the action is precisely $-i\sigma^a k^a$ where k^a is the magnetic charge in the U(1) subgroup corresponding to the Cartan generator H^a. The Dirac quantization condition implies that each magnetic charge is quantized in integer units: $k^a \in \mathbb{Z}$, and so the path integral is invariant under shifts of the form $\sigma^a \to \sigma^a + 2\pi n^a$ with $n^a \in \mathbb{Z}$. Thus, like ϕ^a, each σ^a is a periodic variable with period 2π.

In the simplest case of gauge group SU(2), we have two periodic scalars $\phi = \phi_1 - \phi_2$ and $\sigma = \sigma_1 - \sigma_2$. After dividing out by the Weyl group of SU(2), the classical Coulomb branch is T^2/Z_2 where T^2 is the two dimensional torus parametrized by ϕ and σ. The Z_2 gauge symmetry has four fixed points at $(\phi, \sigma) = (0, 0)$, $(\pi, 0)$, $(0, \pi)$ and (π, π). The fixed point at the origin $(\phi, \sigma) = (0, 0)$ is special because, at least in the classical theory, the non-abelian gauge symmetry is restored at this point. This means, in particular, that there are additional massless degrees of freedom and a naive description in terms of the fields ϕ and σ should break down. In fact, the three-dimensional duality transformation by which σ was introduced is only possible when the low-energy theory is abelian and hence cannot be implemented at this point. We will find below that the origin continues to have a distinguished role in the quantum theory. In contrast, the other three fixed points can be taken at face value as orbifold points of the Z_2 gauge symmetry: no additional massless degrees of freedom appear at these points.

The low-effective energy effective action on the classical Coulomb branch of the SU(2) theory is obtained by compactification of a four-dimensional U(1) gauge theory with gauge coupling g^2 and vacuum angle θ [5]. The resulting three-dimensional theory is a non-linear σ-model with target space T^2/Z_2. The bosonic terms in the action are

$$S_{\text{eff}} = \frac{1}{16\pi R} \int d^3x \left(\frac{4\pi}{g^2}\,\partial_\mu\phi\right)^2 + \left(\partial_\mu\sigma + \frac{\theta}{2\pi}\,\partial_\mu\phi\right)^2 . \tag{1.4}$$

This lagrangean has a unique extension with $\mathcal{N} = 1$ supersymmetry. To make this explicit we combine the real scalars ϕ and σ to form a complex scalar $X = -i(\sigma + \tau\phi)$ where $\tau = 4\pi i/g^2 + \theta/2\pi$ is the usual complexified coupling. The action becomes

$$S_{\text{eff}} = \frac{1}{16\pi R} \int d^3x\, \frac{1}{\text{Im}\tau}\, \partial_\mu X\, \partial_\mu \bar{X} . \tag{1.5}$$

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This is the standard action for a non-linear σ-model with a complex torus E as target. E has periods $2\omega_1 = 2\pi i$ and $2\omega_2 = 2\pi i\tau$, and area $A = 1/16\pi R$. The $\mathcal{N} = 1$ supersymmetric extension of the action is obtained by promoting X to a chiral superfield. The resulting lagrangean can be written as a D-term in $\mathcal{N} = 1$ superspace

$$\mathcal{L}_{\rm eff} = \int d^2\theta\, d^2\bar{\theta}\, \mathcal{K}_{\rm cl}\big[X, \bar{X}\big] \tag{1.6}$$

with the Kähler potential $\mathcal{K}_{\rm cl} = X\bar{X}/(16\pi R{\rm Im}\tau)$.

The above discussion generalizes easily to gauge group SU(N) with $N \geq 2$. In this case we have N chiral superfields $X^a = -i(\sigma^a + \tau\phi^a)$. For a generic point on the classical Coulomb branch the unbroken gauge group is $\mathrm{U}(1)^{N-1}$. On submanifolds where one or more of the X^a coincide, a non-abelian gauge symmetry is restored. As for SU(2), the scalar fields naturally take values on a complex torus E with complex structure parameter τ. We must also divide by Weyl gauge transformations which permute the N Cartan generators. Thus the U(N) Coulomb branch is the symmetric product $\mathrm{Sym}^N(E) = E^N/S^N$ where S^N is the permutation group of N objects. As above, we restrict to gauge group SU(N) by imposing $\sum_{a=1}^{N} X^a = 0$. The low-energy effective action on the Coulomb branch is therefore $\mathcal{N} = 1$ supersymmetric σ-model with target E^{N-1}/S^N. The Kähler metric on the target manifold is obtained from the Kähler potential $\mathcal{K}_{\rm cl} = \sum_{a=1}^{N} X^a\bar{X}^a/(16\pi R{\rm Im}\tau)$.

At a generic point on the classical Coulomb branch the chiral superfields X^a are the only massless degrees of freedom. As discussed above, the only exceptional points are those at which a non-abelian subgroup of the gauge group is restored. Away from these points all other fields in the theory acquire masses via the Higgs mechanism. In this case we expect to be able integrate out the massive degrees of freedom and derive an effective action for the X^a. As we must preserve $\mathcal{N} = 1$ supersymmetry, the most general possible action, including terms with at most two derivatives or four fermions is

$$\mathcal{L}_{\rm eff} = \int d^2\theta\, d^2\bar{\theta}\, \mathcal{K}\big[X^a, \bar{X}^a\big] + \int d^2\theta\, \mathcal{W}\big(X^a\big) + \int d^2\bar{\theta}\, \bar{\mathcal{W}}\big(\bar{X}^a\big)\,, \tag{1.7}$$

where $\mathcal{W}(X^a)$ is a holomorphic superpotential which lifts all or part of the vacuum degeneracy. The supersymmetric vacua of the theory are then determined by the stationary points of the superpotential. In general, this description of the low-energy physics is self-consistent only if the resulting vacua are located away from the point $X = 0$.

In the classical theory the Kähler potential is given by $\mathcal{K}_{\rm cl}$ defined above and the superpotential vanishes identically. Quantum corrections will modify both the D-term and the F-term in the superspace action. In general the exact effective action will depend in a complicated way on the parameters of the theory, including τ, R and m_i, with $i = 1, 2, 3$. As usual, corrections to the Kähler potential are relatively unconstrained and therefore hard to determine. In contrast, the superpotential is constrained by holomorphy in the fields X^a. By standard arguments [13], the superpotential is also holomorphic in the complexified coupling τ and the masses m_i. In many similar situations in three and four dimensions, global symmetries provide further constraints on $\mathcal{W}$. The present case is somewhat different as, for generic values of the masses m_i, the theory has no unbroken global symmetries. However, as we discuss below, we have instead a different constraint which is special to theories with compact dimensions: each chiral superfield X^a is a complex variable with two periods. In section 3, the consequences of double-periodicity and holomorphy will be exploited to determine the exact superpotential. However, as a preliminary, we will first discuss the quantum corrections to the superpotential in the weak-coupling limit $\tau \to i\infty$.

2. Semiclassical analysis

In the weak coupling limit, the path integral is dominated by field configurations of minimum action in each topological sector. In the following discussion we will restrict our attention to the euclidean theory with gauge group SU(2). Gauge field configurations in the compactified theory are labelled by two distinct kinds of topological charge [14]. The first is the Pontryagin number carried by instantons in four-dimensions

$$p = \frac{1}{8\pi^2} \int_{\mathbb{R}^3 \times S^1} \mathrm{Tr}[F \wedge F] \,. \tag{2.1}$$

The 4D instanton number appears in the microscopic action as the term $-ip\theta$. For each value of p, the action is minimized by configurations which solve the (anti-)self-dual Yang-Mills equation. An important feature of the compactified theory which differs from the theory on $\mathbb{R}^4$ is that p is not quantized in integer units. However, for each integer value of p, the theory has solutions with action $S_{\rm cl} = 8\pi^2|p|/g^2 - ip\theta$. Thus we have $S_{\rm cl} = 2\pi i p\tau$ for $p > 0$ and $S_{\rm cl} = -2\pi i p\bar{\tau}$ for $p < 0$. When the instanton scale size is much less than the compactification radius, these solutions are close to their counterparts on $\mathbb{R}^4$, thus we will call them four-dimensional (4D) instantons.

In addition to the usual four-dimensional instantons, the compactified theory also has finite action configurations which carry three-dimensional magnetic charge

$$k = \frac{1}{8\pi} \int_{\mathbb{R}^3} \vec{\nabla} \cdot \vec{B} \in \mathbb{Z} \,, \tag{2.2}$$

where $B_i = \varepsilon_{ijk} F^{jk}/2$ is the abelian magnetic field of the low-energy theory. A useful way to think about these configurations is to introduce a fictitious fifth dimension.

As our starting point is a euclidean theory with four space-like dimensions, we will interpret the new coordinate as time. With this interpretation, the Wilson lines play the role of the adjoint Higgs fields in the Bogomol'nyi equation and the theory has static BPS monopole solutions. As usual, quantization of magnetic charge can be understood either from Dirac's argument or by identifying k with an element of the non-trivial homotopy group $\pi_2[\mathrm{SU}(2)/\mathrm{U}(1)] = Z$. As these configurations do not depend either on the time or on the coordinate of the compactified spatial dimension, we will refer to them as three-dimensional (3D) instantons. In addition to the integer-valued magnetic charge k, these 3D instantons also carry fractional Pontryagin number $p = \phi/2\pi$ [15]. The magnetic charge appears in the action as the term $-ik\sigma$ while, as above, 4D instanton number appears as the term $-ip\theta$. Including these term the action of a 3D instanton can be written as $S_{\rm 3D} = -i\tau\phi|k| + ik\sigma$. Thus we have $S_{\rm 3D} = kX$ for $k > 0$, and $S_{\rm 3D} = k\bar{X}$ for $k < 0$.

As discussed in the previous section, $\mathcal{N} = 1$ supersymmetry permits a non-zero holomorphic superpotential $\mathcal{W}(X)$. Although the superpotential is absent in the classical theory it can be generated by quantum effects [16]. The leading semiclassical contribution of a 4D instanton (for $p > 0$) is $\exp(-S_{\rm 4D}) = \exp(2\pi ip\tau) = q^p$ while that of a 3D instanton (for $k > 0$) is $\exp(-S_{\rm 3D}) = \exp(-kX)$. As these contributions are holomorphic in τ and X respectively, both kinds of instantons potentially contribute to the superpotential at leading semiclassical order. In order to decide whether these contributions are nonzero, we need to examine the fermion zero modes of each type of instanton. In particular, the superpotential contains couplings which are bilinear in Weyl fermions of one four-dimensional chirality. To contribute, an instanton must therefore have exactly two zero modes of the same chirality.

Before discussing each type of instanton individualy, it is possible to make some comments which apply equally to 3D instantons, 4D instantons and also to some more exotic solutions which we will meet below. As both 3D and 4D instantons satisfy the self-dual Yang-Mills equation, both are invariant under supercharges of one four-dimensional chirality. However, the action of the remaining supercharges on the instanton is nontrivial and, by Goldstone's theorem, each generates a fermion zero mode. Thus in a theory with $\mathcal{N} = 1$ supersymmetry (i.e., four supercharges), the existence of the required zero modes is guarenteed for both types of instanton. However, for any self-dual configuration, index theorems suggest the existence a much larger number of zero modes. Specifically, they suggest a number of zero modes which grows linearly with the topological charge. If these zero modes were really present, instanton contributions to the superpotential would vanish except, perhaps, in the sector of lowest topological charge. In fact, in a wide range of cases [17, 18, 19], we know that almost all of these zero modes are lifted by quantum effects. Typically this is possible because the relevant index theorems do not take into account Yukawa couplings between fermions and the scalar fields of the theory. The only exceptions are zero modes whose existence is guarenteed by an unbroken symmetry of the theory. In particular, this is the case for the two zero modes corresponding to the action of the supercharges on the instanton.

The 3D instantons which carry magnetic charge k provide a concrete illustration of the general discussion given above. As these configurations are essentially BPS monopoles, the Callias index theorem predicts the existence of $2k$ zero modes for each species of massless adjoint Weyl fermion. It is convenient to start by considering the case when the three masses m_i are set to zero. In this case, we have an $\mathcal{N} = 4$ supersymmetric theory with four massless species of Weyl fermion and we find a total of $8k$ zero modes. As we now have sixteen supercharges, eight of these modes are Goldstone modes for the half of the supersymmetry algebra which acts non-trivially on self-dual configurations. These modes are protected from quantum corrections by supersymmetry and therfore remain as exact zero modes in the $\mathcal{N} = 4$ quantum theory. The remaining modes are not protected by any symmetry and can thus be lifted in the quantum effects. In fact, at least in the three-dimensional limit, this lifting was demonstrated explicitly in [19]. The lifting occurs in a by now familiar way: the classical instanton action acquires a four-fermion term which couples to the Riemann tensor on the monopole moduli space. Because of this effect, 3D instantons have eight exact zero modes and therefore contribute to an eight fermion vertex in the effective action for each value of k. If we break $\mathcal{N} = 4$ supersymmetry down to $\mathcal{N} = 1$ by reintroducing non-zero masses for the three adjoint chiral multiplets, six of the eight zero modes are lifted and the instantons can contribute directly to the superpotential. The leading semiclassical contribution of 3D instantons of positive magnetic charge k, therefore has the form $m_1 m_2 m_3 \exp(-kX)$.

The form of the contribution of 3D instantons to the superpotential immediately presents us with a puzzle. In the classical theory, ϕ and σ are both periodic variables with period 2π. At weak coupling, we might expect the action to respect both these classical periodicities. This point will be discussed further below. For the moment it suffices to note that, as we have $X = -i(\sigma + \tau\phi)$, the instanton-generated term in the superpotential is manifestly periodic in σ. However it is certainly not periodic in ϕ: if we shift the Wilson line ϕ by 2π, $\exp(-X)$ picks up a factor of $q = \exp(2\pi i\tau)$, the factor associated with a 4D instanton! In fact this reflects a phenomenon which has recently been studied in detail by Lee and Yi [15]. Recall that a single BPS monopole can be thought of as a radial kink in three-dimensional space which interpolates between a Coulomb phase vacuum at spatial infinity and the symmetric vacuum at the origin. Similarly the 3D instanton described above interpolates between a vacuum with non-zero Wilson line $\phi(r = \infty) = \phi$ at radial infinity and the symmetric vacuum with $\phi(r = 0) = 0$ at the origin. However as ϕ is periodic, we could equally well choose a solution of the Bogomol'nyi equations which has $\phi(r = \infty) = \phi + 2n\pi$ for any integer $n \geq 1$. The new configuration is just the standard Prasad-Sommerfeld solution appropriately rescaled to match the shifted value of the asymptotic Wilson line. These solutions have Pontryagin number $n + \phi/2\pi$ and euclidean action

$$S_+^{(n)} = \frac{4\pi}{g^2}(\phi + 2\pi n) - i\sigma - i\theta(\phi + 2\pi n) = X - 2\pi i\tau n \tag{2.3}$$

for each integer $n \geq 1$.

So far we have an infinite tower of 3D instantons, each with magnetic charge $k = 1$, labelled by a positive integer n. The action of each member of the tower is related to that of the ordinary 3D instanton by the replacement $\phi \to \phi + 2\pi n$. The idea now is that we can potentially obtain a superpotential which is periodic in ϕ by summing over sectors of different n. However, for this to have any chance of working, we also need to find a solution corresponding to the replacement $\phi \to \phi - 2n\pi$ for each n. In fact the necessary solutions are also discussed in [15] and the surprise is that they have negative magnetic charge $k = -1$. Importantly these are solutions of the same Bogomol'nyi equation $B_i = +D_i\Phi$ as the ordinary $k = +1$ monopole, and thus are *not* related to the anti-monopole which satisfies $B_i = -D_i\Phi$. This possiblity arises because of the periodicity of the Wilson line which allows us to consider Higgs fields which have a positive value at the origin and decrease in the radial direction. By changing the sign of the radial derivative of Φ we then change the sign on the right-hand side of the Bogomol'nyi equation. In this way we obtain an infinite tower of $k = -1$ solutions with action

$$S_-^{(n)} = \frac{4\pi}{g^2}(2\pi n - \phi) - i\sigma - i\theta(2\pi n - \phi) = -X - 2\pi i\tau n \tag{2.4}$$

for each integer $n \geq 1$. The fact that these configurations have negative magnetic charge yet satisfy the Bogomol'nyi equation with positive sign is reflected in the fact that the action is holomorphic in X rather than anti-holomorphic as would be the case for the action of an anti-monopole.

The above discussion should generalize in a straightforward way to higher magnetic charge: in addition to the usual charge-k 3D instanton with action kX we find two infinite towers of solutions with action $kS_+^{(n)}$ and $kS_-^{(n)}$ respectively, $n \geq 1$, with magnetic charges $+k$ and $-k$. Also, the analysis of the fermion zero modes given above for the ordinary charge-k 3D instanton solution applies equaly to each member in the tower: all zero modes not protected by supersymmetry should be lifted by quantum effects. Including the effects of non-zero masses, we find that configurations in each sector have only two exact zero modes and can therefore contribute to the superpotential.

Finally, we still have to discuss the contribution of ordinary 4D instantons. As discussed in [15], 4D instantons have a very natural description in terms of the configurations described above. Consider for example a configuration corresponding to an ordinary 3D instanton with $k = +1$ together with the lowest member ($n = 1$) of the $k = -1$ tower at large spatial separation. A priori this is only an approximate solution of the field equations. However, because of the sign flip in the Bogomol'nyi equation, the forces between the two constituent objects cancel and it is plausible that a corresponding exact solution of the self-dual Yang-Mills equations exists. As the action of this configuration is $2\pi i\tau$, this solution is naturally identified with a single Yang-Mills instanton on $\mathbb{R}^3 \times S^1$. According to the Atiyah-Singer index theorem a single SU(2) instanton has four fermion zero modes per species of massless Weyl fermion. In the presence of non-zero masses m_i, we have a single massless gluino and thus four zero modes. This counting agrees with the fact that the two constituent 3D instantons have two zero modes each. However, as above, only two zero modes are protected by supersymmetry and we expect the two remaining modes to be lifted. In the four-dimensional theory, the extra modes are generated by superconformal transformations in the instanton background and are protected if this symmetry remains unbroken. In the present case, even when the masses m_i are zero, superconformal invariance is broken by compactification on $\mathbb{R}^3 \times S^1$.

Similar considerations indicate that 4D instantons can contribute to the superpotential for each topological charge $p > 0$. These configurations have magnetic charge zero and so they simply contribute a constant term proportional to $m_1 m_2 m_3 q^p$. As only the derivatives of the superpotential with respect to the chiral field X appear in the low energy effective action, one might wonder whether such a term has any significance. In fact these terms contribute directly to the Greens functions of chiral operators. For example, the gluino condensate is given by the standard relation, $\langle \lambda\lambda \rangle = \partial_\tau \mathcal{W}$, which is obtained by promoting τ to a background chiral superfield [20].

Putting everything together we can write out the most general possible superpotential which can generated at leading semiclassical order

$$\begin{aligned}\mathcal{W} = m_1 m_2 m_3 \sum_{p=1}^{\infty} a_p q^p + m_1 m_2 m_3 \sum_{k=1}^{\infty} b_k \exp(-kX) + \\ + m_1 m_2 m_3 \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} c_{k,n} q^{kn} \exp(-kX) + d_{k,n} q^{kn} \exp(+kX)\,. \qquad (2.5)\end{aligned}$$

In fact, we can immediately rule out any perturbative corrections to this result by noting that dependence on the vacuum angle θ cannot arise in perturbation theory. By holomorphy in τ, any powers g^2 appearing in the superpotential are necessarily accompanied by θ-dependence and are thus forbidden. This argument applies equally to perturbation theory in the vacuum sector or in an instanton background. The exact superpotential is therefore completely determined at leading semiclassical order. This means in particular that the dimensionless coefficients a_p, b_k, $c_{k,n}$ and $d_{k,n}$ do not depend on X or on τ. However, at this point it appears that they can be arbitrary functions of the three dimensionless parameters $m_i R$. In the rest of the paper we present arguments which determine the exact superpotential and, therefore, the corresponding coefficients. In particular we will discover that the superpotential is actually independent of R.

The discussion given above implicitly assumes that the quantum theory has a regime in which the degrees of freedom corresponding to the chiral superfield X are weakly-coupled and a semiclassical analysis makes sense. In such a regime the instanton corrections appearing on the RHS of (2.5) should be exponentially small. Apart from requiring $g^2 \ll 1$, this condition also constrains X. In similar situations in three or four dimensions, we could ensure weak coupling by going to an asymptotic region of the moduli space where the scalar fields have large expectation values. The present case is more subtle because the Coulomb branch is a compact torus. In particular, this is reflected by the presence of terms of order $q^k \exp(+kX)$ in (2.5) which *grow* exponentially as $g^2 \to 0$ if $\phi > 2\pi$. Similarly the terms of order $\exp(-kX)$ grow at weak coupling for $\phi < 0$. However, provided we choose a Wilson line in the range $0 < \phi < 2\pi$, then every term in (2.5) vanishes exponentially as $g^2 \to 0$. In this region of field space, the weak coupling analysis given in this section is reliable. At first sight this restriction on ϕ, seems to contradict the fact that the classical theory is periodic in this variable. As anticipated above, the resolution of this puzzle must be that the periodicity of $\mathcal{W}$ in ϕ is only visible after summing over Lee-Yi copies and multi-instantons. A semiclassical expansion of the superpotential can then be performed for any value of $\phi \neq 2n\pi$ after an appropriate change of variables. In the next section, this idea will be implemented explicitly.

3. The exact superpotential

The superpotential of the SU(2) theory is a holomorphic function on the chiral superfield $X = -i(\sigma + \tau\phi)$. The weak coupling arguments given in the previous section indicate that this function should respect the classical periodicity of the variables ϕ and σ. Thus, $\mathcal{W}(X)$ is a holomorphic function defined on the complex torus E. The superpotential should also be invariant under the Weyl group of SU(2) which acts as $X \to -X$. As $\mathcal{W}(X)$ is a non-constant holomorphic function on a compact domain it must have a singularity somewhere. As usual such a singularity should correspond to a point at which our low-energy description of the theory breaks down due to the presence of extra light degrees of freedom. There is only one such point on the classical moduli space: the origin $X = 0$ where non-abelian gauge symmetry is restored. Thus we will assume that $\mathcal{W}$ has an isolated singularity at this point and no other singularities. In fact we will limit ourselves further by concentrating on the minimal case where the singuarity is a pole, providing numerous checks that the resulting solution is correct. At first sight, the presence of a singularity at the origin in the quantum theory might seem to be in conflict with our understanding of $\mathcal{N} = 2$ theories in four dimensions [4, 2]. In such theories, the classical singularity at the origin of the Coulomb branch is split by strong-coupling effects into two or more singularities where BPS monopoles or dyons become massless. However, in the present context, four-dimensional monopoles correspond to instantons and do not appear as states in the Hilbert space. Thus candidates for the massless states required to explain multiple singularities are absent.

It follows from the above discussion that we are looking for a meromorphic function on the torus E which is even under the action of the Weyl group. It is convenient to introduce an explicit realization of the torus E as the quotient $\mathbb{C}/\Gamma$ where Γ is the lattice

$$\Gamma = \left\{ z = 2m\omega_1 + 2n\omega_2 : (m,n) \in \mathbb{Z}^2 \right\} \tag{3.1}$$

with $\omega_2/\omega_1 = \tau$. Comparing with the results of section 1.2 we have $\omega_1 = i\pi$. A period parallelogram is any set of the form

$$D_\Gamma(z_0) = \left\{ z = z_0 + 2\mu\omega_1 + 2\nu\omega_2 : \mu,\nu \in [0,1) \right\} \tag{3.2}$$

complex $z_0 \in \mathbb{C}$. A meromorphic function on E is the same as a meromorphic doubly-periodic function on the complex plane with periods $2\omega_1$ and $2\omega_2$. These are known as elliptic functions and they are very highly constrained objects (see for example [6]). Some relevant facts are:

(1) Elliptic functions are classified by their *order*, γ, which is equal to the total number of poles in any period parallelogram D_Γ counting by multiplicity (i.e., a double pole contributes $+2$ to γ).

(2) An elliptic function attains an arbitrary complex value γ times in a period parallelogram counting by multiplicity. Thus the total number of zeros in D_Γ is also equal to γ.

(3) Only the constant function has $\gamma = 0$ and there are no functions with $\gamma = 1$. The basic example of a function with $\gamma = 2$ is the Weierstrass function

$$\begin{aligned} \mathcal{P}(X:\Gamma) &= \frac{1}{X^2} + \sum_{W\in\Gamma-\{0\}} \left[\frac{1}{(X-W)^2} - \frac{1}{W^2} \right] = \\ &= \frac{1}{X^2} + \sum_{m,n\neq(0,0)} \left[\frac{1}{(X-2m\omega_1-2n\omega_2)^2} - \frac{1}{(2m\omega_1+2n\omega_2)^2} \right] \end{aligned} \tag{3.3}$$

$\mathcal{P}(X)$ has a double pole at $X=0$. At $\gamma = 2$ there are also the jacobian elliptic functions which have two single poles in D_Γ.

(4) The derivative of an elliptic function is an elliptic function with the same periods. In fact $\mathcal{P}'(X)$ is related to $\mathcal{P}(X)$ by the differential equation

$$\begin{aligned} \left(\frac{d\mathcal{P}}{dX}\right)^2 &= 4\mathcal{P}^3 - g_2(\tau)\mathcal{P} - g_3(\tau) = \\ &= \big(\mathcal{P} - e_1(\tau)\big)\big(\mathcal{P} - e_2(\tau)\big)\big(\mathcal{P} - e_3(\tau)\big), \end{aligned} \tag{3.4}$$

where

$$g_2 = 60 \sum_{(m,n)\neq(0,0)} \frac{1}{(2m\omega_1 + 2n\omega_2)^4}\,,$$
$$g_3 = 140 \sum_{(m,n)\neq(0,0)} \frac{1}{(2m\omega_1 + 2n\omega_2)^6}\,. \tag{3.5}$$

(5) The Weierstrass function is even: $\mathcal{P}(X) = \mathcal{P}(-X)$. Any even elliptic function can be expressed as a rational function of $\mathcal{P}(X)$. Correspondingly any odd elliptic function can be expressed as $\mathcal{P}'(X)$ times a rational function of $\mathcal{P}(X)$.

As the Weyl group action is $X \to -X$, $\mathcal{W}(X)$ must be an even elliptic function and thus, from (5), a rational function of $\mathcal{P}(X)$. In fact, because the only singularity of the superpotential is at the origin, $\mathcal{W}$ must be a polynomial in $\mathcal{P}(X)$. The order, $\gamma_{\mathcal{W}}$, of $\mathcal{W}(X)$ is then twice the degree of this polynomial. From (4), we also learn that the derivative of $\mathcal{W}$ is an odd elliptic function of order $\gamma_{\mathcal{W}}+1$. To pin down the exact function we seek, we must provide one more piece of physics input. Recall that the supersymmetric vacua of the theory correspond to the zeros of $\partial\mathcal{W}/\partial X$. Simple zeros correspond to massive vacua while a higher order zero yields a vacuum with massless particles. Following [5], we will assume that the physics varies smoothly with R and correspondingly we expect the number and type of vacua to be the same as in the four-dimensional theory. According to the arguments in section 1.1, the four-dimensional theory with gauge group SU(2) and three massive chiral multiplets has three vacua: two coming from the confining phase where the low energy theory is $\mathcal{N}=1$ SYM and an extra vacuum in which the gauge group is completely broken. In each of these vacua, the theory has a mass gap. Three massive vacua means that $\gamma_{\mathcal{W}}+1=3$. Finally, as there is only one elliptic function of order two with a double pole at the origin we have the result,[2]

$$\mathcal{W}(X) = m_1 m_2 m_3 \big(\mathcal{P}(X) + C(\tau)\big)\,. \tag{3.6}$$

Importantly, we have obtained a superpotential which is independent of R. Strictly speaking we have not determined the overall normalization of (3.6). This will ultimately be determined (and shown to be independent of R) in the next section. Note also that, so far, we have only determined the superpotential up to the addititive constant $C(\tau)$. This ambiguity will be discussed further below.

From the differential equation (3.4) we see that, as required, $\partial\mathcal{W}/\partial X$ has three simple zeros and that the corresponding critical values $\mathcal{W}$ are $e_1(\tau)$, $e_2(\tau)$ and $e_3(\tau)$.

[2]Two dimensional Landau-Ginzburg theories with this superpotential have been considered before. See [21] and references therein.

In fact these values are attained at the half-lattice points $X = \omega_1 = i\pi$ $X = \omega_1 + \omega_2 = i\pi(\tau + 1)$ and $X = \omega_2 = i\pi\tau$ respectively. These are three of the four fixed points of the Weyl group discussed in section 1.2 above. In the remainder of this section we highlight several interesting features of this result:

Modular properties: The lattice Γ is invariant under modular transformations acting on the half-periods as $\omega_2 \to a\omega_2 + b\omega_1$ and $\omega_1 \to c\omega_2 + d\omega_1$ with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. As the function $\mathcal{P}(X)$ only depends on the choice of the lattice Γ it is invariant under these transformations. Recalling that $\tau = 4\pi i/g^2 + \theta/2\pi = \omega_2/\omega_1$ we see that these transformations correspond to the familiar action of S-duality on the complexified coupling of the $\mathcal{N} = 4$ theory, namely $\tau \to (a\tau + b)/(c\tau + d)$. In fact this is not quite the same as the action on the periods defined above because we have $\omega_1 = i\pi$ and $\omega_2 = i\pi\tau$ so only ω_2 transforms. S-duality corresponds to a modular transformation of the lattice accompanied by a rescaling by a factor of $(c\omega_2 + d\omega_1)$ in order to preserve $\omega_1 = i\pi$. Taking this into account we discover that the superpotential $\mathcal{W}$ transforms with modular weight two under S-duality. This means that the F-term effective lagrangean of the theory is invariant under S-duality if we assign the superspace measure $d^2\theta$ modular weight -2.

It is also worth noting that *any* non-zero $C(\tau)$ would spoil these modular properties, or at least introduce unphysical poles for some values of τ. The reason for this is as follows: while $\mathcal{W}$, as function of X and τ, is free to transform with modular weight two, a holomorphic function of τ alone, with no poles in the fundamental domain of $\mathrm{SL}(2, \mathbb{Z})$, cannot. If it were to do so it would be a modular form of weight two of which there are none [22]. Thus modularity of the superpotential uniquely picks $C(\tau) = 0$. The action of S-duality on the critical points of the potential is also interesting. As above, $\mathcal{W}$ transforms with modular weight two, but its critical values $e_i(\tau)$, as functions of τ alone, cannot. The modular properties of the e_i were described in [2]. Each is modular only with respect to a certain subgroup of the modular group. Under the rest of the modular transformations, the e_i transform into each other. This means that S-duality transformations permute the three vacua. This is very natural in the light of the interpretation of these vacua given in [2]. In particular the three vacua are associated with the condensation of elementary quanta, monopoles and dyons respectively. As the S-duality of the $\mathcal{N} = 4$ theory permutes these BPS states it is reasonable that it should do the same to the corresponding vacua of the $\mathcal{N} = 1$ theory.

Semiclassical expansion: In section 2, we argued that the exact superpotential should have a very specific form at weak coupling with semiclassical contributions from 4D instantons, 3D instantons together with two infinite towers of Lee-Yi copies. We can now expand our proposal for the exact superpotential in the semiclassical limit and compare it with these expectations. Expanding the Weierstrass function for $\tau \to i\infty$ we obtain

$$\mathcal{W} = m_1 m_2 m_3 \left[\frac{1}{12} E_2(\tau) + \frac{1}{4} \sum_{n=-\infty}^{+\infty} \frac{1}{\sinh^2((X + 2\pi i n\tau)/2)} \right] =$$
$$= m_1 m_2 m_3 \sum_{k=1}^{\infty} k \exp(-kX) + m_1 m_2 m_3 \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} k q^{kn} \Big[\exp(-kX) + \exp(+kX) - 2 \Big], \tag{3.7}$$

where $E_2(\tau)$ is the regulated Eisenstein series of modular weight two [22]. Comparing with (2.5) we deduce that $a_p = \sum_{d|p} d$ and $b_k = c_{k,n} = d_{k,n} = k$. In hindsight, we could have deduced the n-independence of the coefficients $c_{k,n}$ and $d_{k,n}$ simply by requiring 2π-periodicity of $\mathcal{W}$ in the Wilson line ϕ. These predictions could be tested against first-principles field theory calculations.

Two limiting cases: We will now consider the limit in which we obtain a three dimensional gauge theory. This corresponds to taking the limit $R \to 0$, $\tau \to i\infty$ while keeping the three dimensional gauge coupling $e^2 = g^2/R$ fixed. In this limit, the period of the torus associated with the Wilson line goes to infinity while the period corresponding to the dual photon σ stays fixed. Hence, both the 4D instantons and the towers of Lee-Yi copies have infinite action in this limit and the corresponding contributions to $\mathcal{W}$ vanish giving

$$\mathcal{W} = m_1 m_2 m_3 \frac{1}{\sinh^2(X/2)}\,. \tag{3.8}$$

This is a prediction for the superpotential of a theory with $\mathcal{N} = 1$ supersymmetry which can be obtained as a massive deformation of the three-dimensional SO(8) invariant conformal field theory with sixteen supercharges. Note that the superpotential has a double pole at the SO(8) invariant point $X = 0$ where the Coulomb branch description breaks down. The theory has a single supersymmetric vacuum at $X = i\pi$ which corresponds to an orbifold fixed points at which the corresponding theory with sixteen supercharges is believed to be free [24]. In the weak coupling limit we can re-expand the superpotential and see that it includes an infinite series of 3D instanton corrections closely related to the 3D instanton series summed in [19] for the theory with sixteen supercharges.

Another interesting limit is obtained by keeping R finite but decoupling the three chiral multiplets. This gives four-dimensional $\mathcal{N} = 1$ SYM theory with gauge group SU(2) compactified on a circle of radius R. Thus we take the limit $m_i \to \infty$ and $\tau \to i\infty$ with $\Lambda = (m_1 m_2 m_3)^{1/3} q^{1/6}$ held fixed. Naively, the superpotential appears to be divergent in this case. In fact to obtain the correct answer we need to redefine our fields according to $X = Y - \pi i\tau$. In terms of Y, we see that precisely two terms in (3.7) survive in this limit: the contribution from the 3D instanton with magnetic charge $k=1$, which goes like $q^{1/2}\exp(-Y)$, and the $n=1$ term in the tower of states with magnetic charge -1 which goes like $q^{1/2}\exp(+Y)$. Thus we find

$$\mathcal{W} = \Lambda^3\Big[\exp(-Y) + \exp(+Y)\Big]\,. \tag{3.9}$$

This agrees with the result obtained in [5, 23], which has also recently been derived from first principles in [9]. The fact that only two topological sectors contribute in this limit is due to a global symmetry of the minimal $\mathcal{N}=1$ theory which is broken explicitly in the full theory by couplings of the adjoint matter multiplets. The theory has two supersymmetric vacua, located at $Y=\pm i\pi$, which agrees with the Witten index of $\mathcal{N}=1$ SUSY Yang-Mills with gauge group SU(2).

Note that the two limits considered above are complimentary in a certain sense. Specifically, we started from a Coulomb branch T^2/Z_2 with four Z_2 fixed points of which three correspond to massive vacua. Two of these vacua correspond to the confining phase of the four-dimensional theory while one vacuum corresponds to the Higgs phase. The remaining fixed point is a singular point of the superpotential. In the weak coupling limit, $\tau\to\infty$, one period of the torus diverges and the Coulomb branch becomes an infinite cylinder. However, which points on the torus are sent to infinity depends on how X, m_i and R scale in this limit. The three-dimensional limit considered above corresponds to blowing up the region which contains the Higgs vacuum and the singularity. This sends the two confinement vacua to infinity. In contrast, the limit which yields $\mathcal{N}=1$ SYM on $\mathbb{R}^3\times S^1$ corresponds to blowing up the neighbourhood of the two confining vacua which sends the Higgs vacuum and the singular point to infinity.

Generalisation to gauge group SU(N): The results described above have a fairly obvious generalization to gauge group SU(N). In this section, rather than giving detailed arguments, I will motivate an educated guess for the exact superpotential. In subsequent sections I will present a detailed check that the answer is the correct one.

For gauge group SU(N), we have N chiral superfields X^a $a=1,2,\ldots,N$ with $\sum_{a=1}^N X^a = 0$ and the classical Coulomb branch is E^{N-1}/S^N. The low energy gauge group is $\mathrm{U}(1)^N$ and thus we have $N-1$ species of 3D instantons which correspond to magnetic monopoles associated with each simple root of SU(N). For $a=1,\ldots,N-1$ these contribute terms of order $m_1m_2m_3\exp(X^a-X^{a+1})$ to the superpotential [25, 23]. Each of the fundamental monopoles will have a towers of Lee-Yi copies as well as a series of multi-monopole corrections which must be summed over. Applying the arguments of the previous sections to constrain the the holomorphic dependence on each of the complex variables X^a-X^{a+1}, we can argue that the summation must convert each exponential to the corresponding doubly periodic function $\mathcal{P}(X^a-X^{a+1})$.

However there is also a new feature which enters for $N>2$. In particular we must also consider the contribution of monoples embedded in SU(2) subgroups which correspond to non-simple roots of SU(N). The relevant configurations are ones which include fundamental monopoles charged under two different magnetic U(1)'s. Index theorems indicate that these configurations have more fermion zero modes than a fundamental monopole. If this were true these topological sectors would not contribute to the superpotential. However, once again, the additional zero modes are not protected by any symmetries and therefore will generically be lifted by quantum effects. This effect has been explicity demonstrated in the three-dimensional theory with sixteen supercharges by Fraser and Tong [26]. This suggests that we should find contributions to the superpotential which go like $m_1 m_2 m_3 \exp(X^a - X^b)$ for all $a \neq b$. As above, summation over magnetic charges and the towers of copies should promote each exponential to the doubly periodic function $\mathcal{P}(X^a - X^b)$. Finally the results of [26] suggest that, after taking into account the additional sectors, we should find complete democracy among the roots of SU(N) with no set of simple roots playing a prefered role. Thus the 3D instanton corresponding to each root should contribute to $\mathcal{W}$ with the same overall coefficent. The proposal for the exact superpotential is therefore

$$\mathcal{W}(X) = m_1 m_2 m_3 \sum_{a>b} \mathcal{P}(X_a - X_b)\,, \tag{3.10}$$

where, taking the action of the Weyl group into account, we have restricted the range of the sum to $a > b$ to avoid double-counting.

A detailed test of this proposal for arbitrary N will be given in section 5. However, we can immediately provide a check on the result in the simplest non-trivial case of gauge group SU(3). Recall that the vacua of the corresponding four dimensional theory are in one to one correspondence with the three-dimensional representations of SU(2). According to the discussion in section 1.1, the SU(3) theory has one Higgs phase vacuum corresponding to the irreducible representation, three confining vacua corresponding to the trivial representation and a single Coulomb phase vacuum corresponding to the three-dimensional representation of SU(2) obtained by adding the trivial and fundamental representations. As our analysis holds for all R, each of these vacua should show up as a stationary point of the proposed superpotential. Thus

$$\mathcal{W} = m_1 m_2 m_3 \Big(\mathcal{P}(X_1 - X_2) + \mathcal{P}(X_2 - X_3) + \mathcal{P}(X_3 - X_1)\Big) \tag{3.11}$$

should have five stationary points, one of them massless. This means that the coupled equations $\partial\mathcal{W}/\partial X_a$, $a = 1, 2, 3$ must therefore have five solutions modulo lattice translations and the action of the Weyl group. Note that the hessian matrix $H = \partial^2\mathcal{W}/\partial X_a \partial X_b$, will always have one zero eigenvalue corresponding to shifts in $X_1 + X_2 + X_3$ which is projected out the condition $X_1 + X_2 + X_3 = 0$ for gauge group SU(N). In the four predicted massive vacua there must be no other zero modes. In contrast, in the predicted massless vacuum, H should have at least one additional zero eigenvalue.

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To check this we must solve the equations

$$\begin{aligned}
&\mathcal{P}'(X_1 - X_2) - \mathcal{P}'(X_3 - X_1) = 0\,,\\
&\mathcal{P}'(X_2 - X_3) - \mathcal{P}'(X_1 - X_2) = 0\,,\\
&\mathcal{P}'(X_3 - X_1) - \mathcal{P}'(X_2 - X_3) = 0\,,
\end{aligned} \tag{3.12}$$

or simply $\mathcal{P}'(X_1 - X_2) = \mathcal{P}'(X_2 - X_3) = \mathcal{P}'(X_3 - X_1)$. Solutions should be counted modulo lattice translations and permutations of X_1, X_2 and X_3. We must also impose the constraint that $X_1 + X_2 + X_3$ should vanish up to a lattice translation. Direct calcuation shows that the required zeros of $\partial\mathcal{W}$ are located at

$$\begin{aligned}
(X_1, X_2, X_3) = &\left(0, \frac{2\omega_1}{3}, \frac{4\omega_1}{3}\right)\\
&\left(0, \frac{2\omega_2}{3}, \frac{4\omega_2}{3}\right)\\
&\left(0, \frac{2}{3}(\omega_1 + \omega_2), \frac{4}{3}(\omega_1 + \omega_2)\right)\\
&\left(0, \frac{2}{3}(\omega_1 + 2\omega_2), \frac{2}{3}(2\omega_1 + \omega_2)\right)\\
&\big(\omega_1, \omega_2, \omega_1 + \omega_2\big)\,.
\end{aligned} \tag{3.13}$$

We find that, for generic values of τ, the hessian matrix has only one zero eigenvalue at the first four solutions in (3.13). It is easily checked that H has one extra zero eigenvalue at the fifth solution listed above. Thus we find exact agreement with the vacuum structure of the four-dimensional theory. It is worth noting that this test would be failed by a superpotential which included only the contributions from 3D instantons corresponding to simple roots of SU(3). Specifically the four massive vacua would be absent in this case.

4. Soft breaking of $\mathcal{N} = 2$ SUSY

In the previous sections we analysed the vacuum structure of $\mathcal{N} = 1$ theories with with non-zero masses m_1, m_2 and m_3 in terms of softly-broken $\mathcal{N} = 4$ supersymmetry. In this section, based on the ideas of [5], we study the same theories from a different, and much more powerful, perspective. If we set $m_1 = 0$ and $m_2 = m_3 = M$, the SU(N) gauge theory studied above reduces to $\mathcal{N} = 2$ supersymmetric Yang-Mills theory with a single adjoint hypermultiplet of mass M. In four dimensions, the exact Coulomb branch of the SU(2) theory is governed by an elliptic curve as described by Seiberg and Witten [2] for gauge group SU(2). The corresponding curve for gauge group SU(N), with $N \geq 2$, was given subsequently by Donagi and Witten [3]. The four-dimensional versions of the $\mathcal{N} = 1$ theories studied above can then be realized by re-introducing a non-zero mass $m_1 = \epsilon$ for the remaining massless chiral multiplet. In this approach, there is an elegant correspondence between the massive vacua of the $\mathcal{N} = 1$ theory and the singular points on the Coulomb branch of the underlying $\mathcal{N} = 2$ theory. In fact, in the next section, we will show that the stationary points of our proposed superpotential reproduce this correspondence exactly.

In a parallel development, Seiberg and Witten have also described how this analysis can be extended to apply to $\mathcal{N} = 2$ theories on $\mathbb{R}^3 \times S^1$. We will begin this section by reviewing the key points of their analysis in the simplest case of $\mathcal{N} = 2$ supersymmetric Yang-Mills theory with gauge group SU(2). The discussion will be very brief and we refer the reader to [5] for further details. After this review, we will extend this analysis to the mass deformed $\mathcal{N} = 4$ theories discussed in the previous sections. We will find that this yields an extremely simple derivation of the main results of this paper.

After compactification to $D = 3$ on a circle of radius R, $\mathcal{N} = 2$ SUSY Yang-Mills with gauge group SU(2) has a Coulomb branch $\mathcal{M}$ of complex dimension two. One complex dimension corresponds to the adjoint scalar, $u = \langle \mathrm{Tr}\Phi^2 \rangle$, which parametrizes the Coulomb branch of the four-dimensional theory. The other dimension is essentially the complex superfield X introduced above, whose real and imaginary parts (for $\theta = 0$) are the Wilson line and dual photon respectively. As the gauge fields have been eliminated in favour of scalars, the low-energy effective action is a three-dimensional non-linear σ-model with target $\mathcal{M}$. In order to preserve $\mathcal{N} = 2$ SUSY (in the four-dimensional convention), $\mathcal{M}$ must be hyper-Kähler. In general the hyper-Kähler metric will depend on R in a complicated way. However, the key result of [5] is that $\mathcal{M}$ has a distinguished complex structure which is independent of R. In particular, in terms of coordinates which are holomorphic with respect to the distinguished complex structure, $\mathcal{M}$ is defined by the complex equation

$$y^2 = x^3 - x^2 u + \Lambda^4 x \,, \tag{4.1}$$

where Λ is the dynamical scale of the four-dimensional theory. This is exactly the same equation as that of the elliptic curve which governs the Coulomb branch of the four-dimensional theory. Indeed, near four-dimensions, this can be interepreted as an elliptic fibration of the u-plane, with the Seiberg-Witten curve of the $D = 4$ theory as the fibre.

The distinguished complex structure picks out an $\mathcal{N} = 1$ subalgebra of the $\mathcal{N} = 2$ supersymmetry algebra. In simple terms this just corresponds to taking the holomorphic coordinates of (4.1) and promoting them to $\mathcal{N} = 1$ chiral superfields. As we will eventually break $\mathcal{N} = 2$ supersymmetry down to this $\mathcal{N} = 1$ subalgebra it is useful to determine the low-energy effective lagrangean in a manifestly $\mathcal{N} = 1$ supersymmetric form. Fortunately, to determine the vacuum structure we will not need to know the whole effective action, only the F-term part. Of course, if we knew the Kähler potential for the hyper-Kähler metric in terms of two unconstrained chiral superfields we could, in principle, write down the $\mathcal{N}=2$ σ-model action as a D-term in $\mathcal{N}=1$ superspace. In this case there would be no F-term. In fact, we have instead a description of the target space involving three chiral superfields, x, y and u, subject to the holomorphic constraint (4.1). The constraint can be imposed by introducing a fourth chiral superfield λ as a lagrangean multipler and considering superpotential

$$\mathcal{W}=\lambda\Big(y^2-x^3+x^2u-\Lambda^4x\Big)\,. \tag{4.2}$$

In order to break $\mathcal{N}=2$ SUSY down to the $\mathcal{N}=1$ subalgebra which is manifest in (4.2), we now introduce a superspace mass term, $\Delta\mathcal{W}=\epsilon u$, for the adjoint chiral multiplet. Making the convenient change of variables, $x-u=v$, $x=\Lambda^4\tilde{x}$ and $y=\Lambda^4\tilde{y}$ the superpotential becomes

$$\mathcal{W}+\Delta\mathcal{W}=\lambda\Lambda^8\Big(\tilde{y}^2-\tilde{x}^3+\tilde{x}^2u-\Lambda^4\tilde{x}\Big)+\epsilon\big(\Lambda^4\tilde{x}-v\big)\,. \tag{4.3}$$

We can now solve the F-term equations

$$\frac{\partial\mathcal{W}}{\partial\lambda}=\frac{\partial\mathcal{W}}{\partial\tilde{y}}=\frac{\partial\mathcal{W}}{\partial v}=0 \tag{4.4}$$

with $\tilde{y}=0$, $\lambda=-\epsilon/\Lambda^8\tilde{x}^2$ and $v=-1/\tilde{x}$. On integrating out $\tilde{y}$, λ and v we obtain

$$\mathcal{W}=\epsilon\left(\Lambda^4\tilde{x}+\frac{1}{\tilde{x}}\right)=\epsilon\Lambda^2\big(\exp(-Y)+\exp(+Y)\big)\,, \tag{4.5}$$

where we have mapped the complex $\tilde{x}$-plane to the cylinder parameterized by the dimensionless variable Y with $\tilde{x}\Lambda^2=\exp(-Y)$. Thus we have reproduced the superpotential given in [5, 23] for $\mathcal{N}=1$ SYM theory compactified to three dimensions. Importantly the resulting superpotential is independent of R. This is an immediate consequence of Seiberg and Witten's proposal that distinguished complex structure is independent of R.

Now we will follow precisely the same steps starting from the complex curve which describes mass-deformed $\mathcal{N}=4$ supersymmetric Yang-Mills theory with gauge group SU(2). In terms of $\mathcal{N}=2$ supermultiplets, this model differs from the one considered above by the presence of an additional adjoint hypermultiplet of mass M. Although the Coulomb branch of this theory can be described in terms of a single elliptic curve, we will choose an alternative representation which generalizes more readily to SU(N) with $N>2$. According to Donagi and Witten [3], the Coulomb branch of the SU(2) theory is determined by the pair of equations which can be written as

$$y^2=F(x)=4x^3-xg_2(\tau)-g_3(\tau)=4\big(x-e_1(\tau)\big)\big(x-e_2(\tau)\big)\big(x-e_3(\tau)\big) \tag{4.6}$$

and

$$u=\frac{P^2}{2}+M^2x\,. \tag{4.7}$$

The first equation describes a complex torus E with complex structure parameter τ and a single puncture corresponding to the point $x = y = \infty$ on the curve. This is known as the 'bare' spectral curve and, on its own, it describes an $\mathcal{N} = 4$ theory with gauge group U(1). The second equation, which is particularly simple for gauge group SU(2), encodes the extra dependence of the SU(2) theory on mass-parameters and VEVs. To be more precise we should identify points which are related by the action of $\alpha\beta$

$$\alpha : y \to -y\,, \qquad \beta : P \to -P\,. \tag{4.8}$$

Imitating the logic applied above for the theory without matter we expect the Coulomb branch of the four-dimensional theory compactified on a circle down to three dimensions to be a hyper-Kähler manifold $\mathcal{M}$ of complex dimension two with a distinguished complex structure specified by eqs. (4.6) and (4.7). The low energy effective action should be a three-dimensional non-linear σ-model with target $\mathcal{M}$. The σ-model lagrangean can be written in $\mathcal{N} = 1$ superspace in terms of chiral fields corresponding to the holomorphic coordinates x, y, u and t together with two lagrangean multipliers λ_1 and λ_2. Thus, including the $\mathcal{N} = 2$ breaking mass term, we start from the superpotential

$$\mathcal{W} = \lambda_1\big(y^2 - F(x)\big) + \lambda_2\left(\frac{P^2}{2} + M^2 x - u\right) + \epsilon u\,. \tag{4.9}$$

The equations of motion for λ_1 and λ_2 are eqs. (4.6) and (4.7) respectively. Stationarizing the superpotential with respect to x and u gives the additional equations, $\lambda_1 F'(x) = -M^2\lambda_2$, $\lambda_2 = -\epsilon$. Together with eq. (4.7), these two relations can be used to eliminate λ_1 λ_2 and u in favour of x and P. Substituting for these variables, the superpotential reduces to $\mathcal{W} = \epsilon u = \epsilon(P^2/2 + M^2 x)$. However we still have to impose eq. (4.6) which constrains x in terms of y. As mentioned above, eq. (4.6) describes a complex torus in terms of two complex variables x and y. We can now change variables to a single unconstrained complex variable X, which parametrizes the same torus, by setting $x = \mathcal{P}(X)$ and $y = \mathcal{P}'(X)$. eq. (4.6) is obeyed by virtue of the differential eq. (3.4) of section 4. In terms of X the superpotential now reads,

$$\mathcal{W} = \epsilon\left(\frac{P^2}{2} + M^2\mathcal{P}(X)\right) \tag{4.10}$$

and thus, setting P to zero by it equation of motion, we recover the result of the previous section. As in the previous example, the superpotential is independent of R simply because of the R-independence of the distinguished complex structure. This analysis also resolves the puzzle raised in the previous section concerning the singularity at the origin of the Coulomb branch of the $\mathcal{N} = 1$ theory. The origin of the $\mathcal{N} = 1$ Coulomb branch at $X = 0$, corresponds to the point $x = y = \infty$ on the

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bare spectral curve. Thus the singularity is projected to infinity and does not appear on the Coulomb branch of the corresponding $\mathcal{N} = 2$ theory.

We now turn to the more challenging case of gauge group SU(N). The Donagi-Witten solution of the $D = 4$ theory is given in terms of two complex equations. The first is the bare spectral curve which appeared above in the SU(2) case.

$$y^2 = 4x^3 - xg_2(\tau) - g_3(\tau) \tag{4.11}$$

and the second is an N'th order polynomial equation in t with coefficients which are polynomial in x and y and which also encode the dependence on the hypermultiplet mass M and the $N-1$ moduli $u_n = \langle \mathrm{Tr}\Phi^n \rangle$, $n = 2, \ldots, N$, of the four-dimensional Coulomb branch

$$F_N(t, x, y : M, u_n) = 0\,, \tag{4.12}$$

F_N also depends on $N-1$ additional parameters that we can think of as the coordinates on the fibres of a torus bundle over the four-dimensional Coulomb branch. As in the $N = 2$ case the total space of this fibration is a hyper-Kähler manifold $\mathcal{M}$ of complex dimension $2(N-1)$ which we identify as the Coulomb branch of the compactified theory [28].

Equation (4.12) generalizes equation (4.7) of the SU(2) case. Part of the problem here is that the F_N grow rapidly in complexity with N and there is no explicit formula in the general case. However let us review how much information we require to determine the exact superpotential. In the $N = 2$ case we solved eqs. (4.6) and (4.7) eliminating x, y in favour of X and then u in favour of X and P. In particular, X and P provide an unconstrained parametrization of the target space. After solving the two constraint equations in this way, the corresponding F-terms vanish and only the mass term in the superpotential survives: $\mathcal{W} = \epsilon u(X, P)$. Hence the task of finding the superpotential is simply that of expressing the modulus $u = \langle \mathrm{Tr}\Phi^2 \rangle$ in terms of a set of unconstrained holomorphic coordinates on $\mathcal{M}$.

Fortunately, the information we need to accomplish this is provided by correspondence between $\mathcal{N} = 2$ theories and integrable systems [27]. The particular version of this correspondence is the one given in [28], where the Coulomb branch of a compactified $\mathcal{N} = 2$ theory is identified as the complexified phase space of a certain integrable hamiltonian system. For a theory with gauge group SU(N), the dynamical system consists of N particles on the line interacting via two-body forces. The positions and momenta of the N particles in the centre of mass frame are X_a and P_a, $a = 1, 2, \ldots, N$. After implimenting the trivial centre of mass contraint $\sum_{a=1}^{N} X^a = \sum_{a=1}^{N} P^a = 0$, the $N-1$ independent positions and momenta provide the unconstrained holomorphic parametrization of the Coulomb branch which we seek.

The specific integrable system relevant for the $\mathcal{N} = 2$ theory with an adjoint hypermultiplet considered above, is the elliptic Calogero-Moser hamiltonian [11] defined by the two-body interaction potential $V(X) = M^2\mathcal{P}(X)$. The only fact we need here is that the $N-1$ Coulomb branch moduli u_i are identified with the $N-1$ constants of motion of the system (where we are not counting the total linear momentum which we set to zero above). In particular the adjoint mass operator u_2 is identified with the first non-trivial constant of motion which is the Calogero-Moser hamiltonian, $\mathcal{H}$ itself,[3]

$$u_2 = \mathcal{H} = \sum_{a=1}^{N} \frac{P_a^2}{2} + M^2 \sum_{a>b} \mathcal{P}(X_a - X_b)\,. \tag{4.13}$$

Integrating out the momenta P_a, the superpotential becomes

$$\mathcal{W} = \epsilon M^2 \sum_{a>b} \mathcal{P}(X_a - X_b) \tag{4.14}$$

which agrees with the proposal given in section 3.

5. A test for gauge group SU(N)

In this section we will perform a simple test of the proposed superpotential for the SU(N) theory. In particular, we will compare the critical points of the superpotential with the vacua of the corresponding four dimensional theory. In the present case, the vacuum structure of the $\mathcal{N}=1$ theory which is obtained by soft breaking of $\mathcal{N}=2$ supersymmetry has been discussed in detail by Donagi and Witten [3]. As above the four-dimensional Coulomb branch is described by the two complex eqs. (4.11) and (4.12). The first equation describes the bare spectral curve which is a torus E with complex structure parameter τ. As in section 3, this can be represented as the quotient $\mathbb{C}/\Gamma$ where Γ is a lattice in the complex plane: $\Gamma = Z \oplus \tau Z$. As the second equation is an N'th order polynomial equation in t with coefficients which depend on x and y it defines an N-fold cover, $\mathbb{C} \to E$. The four-dimensional Coulomb branch is then (part of) the jacobian manifold of $\mathbb{C}$. Away from the singular points in its moduli space, $\mathbb{C}$ corresponds to a Riemann surface of genus N.

As usual for $\mathcal{N}=2$ theories in four dimensions, the points on the Coulomb branch which survive as massive vacua after breaking to $\mathcal{N}=1$, correspond to points in moduli space where $\mathbb{C}$ develops nodes and reduces to a surface of lower genus [4]. In this case, the relevant singular points are those at which the maximal degeneration occurs and the genus of $\mathbb{C}$ is reduced to one. According to Donagi and Witten, these points are in one to one correspondence with sublattices $\tilde{\Gamma} \subset \Gamma$ of index N. For the present purposes the index just means the number of points

[3]The quickest way to obtain this identification is from the equality demonstrated in [29] between $\mathcal{H}$ and $\partial\mathcal{F}/\partial\tau$, where $\mathcal{F}(a_n, m_i, \tau)$ is the $\mathcal{N}=2$ prepotential written in terms of the 'electric' periods a_n. We then invoke the Matone relation [30, 31, 32, 33] which, for UV finite theories, reads $u_2 = \partial\mathcal{F}/\partial\tau$.

of Γ contained in each period parallelogram of $\tilde{\Gamma}$. A particularly attractive aspect of this correspondence, is that the action of S-duality on the set, $\mathcal{S}$, of massive vacua is manifest. Specifically, the S-duality group is identified with the $\mathrm{SL}(2,\mathbb{Z})$ automorphism group of Γ. In general, $\mathrm{SL}(2,\mathbb{Z})$ tranformations on Γ will preserve the index of a sublattice Γ' but not the sublattice itself. It follows that S-duality permutes the massive vacua. This is very natural, as each massive vacuum of the theory is associated with the condensation of a particular BPS state of the $\mathcal{N}=2$ theory. An action of $\mathrm{SL}(2,\mathbb{Z})$ on the set of massive vacua is then inherited from the action of S-duality on the BPS spectrum.

It is useful to note for each sublattice $\tilde{\Gamma}$ of index N in Γ, the lattice $\tilde{\Gamma}/N$ itself contains Γ as a sublattice of index N. In fact $\tilde{\Gamma}\subset\Gamma$ with index N if and only if $\Gamma\subset\tilde{\Gamma}/N$ with index N. If we now define $\Gamma'=\tilde{\Gamma}/N$ we can give an equivalent formulation of the correspondence. Namely, given the lattice Γ, the massive vacua are in one to one correspondence with lattices Γ' such that Γ is a sublattice of Γ' with index N.

In the previous section we discovered that the bare spectral curve $E=\mathbb{C}/\Gamma$ is naturally identified with the torus on which the Weierstrass function is defined in the superpotential (4.14). In the following, we will demonstrate that $\mathcal{W}$ has a stationary point corresponding to each Γ' which contains Γ as a sublattice of index N. An intuitive way of understanding this result is suggested by the correspondence with the elliptic Calogero-Moser system described above. In particular, we can think of the complex scalars X^a as the locations of N particles on the torus E. The particles interact via a two-body 'potential' $\sim\mathcal{P}(X)$, albeit a complex one. The condition for a supersymmetric vacuum is just a kind of holomorphic version of the equilibrium condition for this N-body system. As the particles interact via complex repulsive 'forces'[4] $\sim\mathcal{P}'(X)$, the natural equilibrium positions are when the particles form a uniform lattice on E so that the 'forces' cancel pairwise. As there are N particles, this defines a lattice Γ' which has Γ as a sublattice of index N.

We will begin by considering the Weierstrass function, $\mathcal{P}(X:\Gamma)$, defined by the lattice $\Gamma=\{z=2m\omega_1+2n\omega_2:(m,n)\in\mathbb{Z}^2\}$. Two points which we will need are:

(1) The derivative of the Weierstrass function can be represented as

$$\begin{aligned}\mathcal{P}'(X:\Gamma) &= -2\sum_{Z\in\Gamma}\frac{1}{(X-Z)^3} = \\ &= -2\sum_{(m,n)\in\mathbb{Z}^2}\frac{1}{(X-2m\omega_1-2n\omega_2)^3}\,.\end{aligned}\tag{5.1}$$

[4]Of course the analogy is a loose one because the 'force' between two particles is not directed along the line in the complex plane joining their positions.

The resulting double series is uniformly convergent for all $X \notin \Gamma$. In particular, this means that the sum does not depend on which order the individual sums over m and n are performed [6].

(2) For any lattice Γ, we know that a point $z = w$ is in Γ if and only the point $z = -w$ is. As the series in question converges, we then have

$$\sum_{Z \in \Gamma - \{0\}} \frac{1}{Z^3} = 0 \,. \tag{5.2}$$

To demonstrate the required stationary points of $\mathcal{W}$, must solve the N equations[5]

$$\frac{\partial \mathcal{W}}{\partial X_b} = -\sum_{a \neq b} \mathcal{P}'(X_a - X_b) = 0 \tag{5.3}$$

for $b = 1, \ldots, N$ modulo lattice translations and the action of the Weyl group. Using (5.1), we have

$$\begin{aligned}
\frac{\partial \mathcal{W}}{\partial X_b} &= 2 \sum_{a \neq b} \sum_{Z \in \Gamma} \frac{1}{(X_a - X_b - Z)^3} = \\
&= 2 \sum_{a=1}^{N} \sum_{Z \in \Gamma - \{0\}} \frac{1}{(X_a - X_b - Z)^3} + 2 \sum_{a \neq b} \frac{1}{(X_a - X_b)^3} = \\
&= \sum_{Y \in S(b)} \frac{1}{Y^3} + \sum_{Y \in T(b)} \frac{1}{Y^3}
\end{aligned} \tag{5.4}$$

with the sets $S(b)$ and $T(b)$ defined as

$$\begin{aligned}
S(b) &= \Big\{ Y = X_a - X_b - Z : Z \in \Gamma - \{0\} : a = 1, 2, \ldots, N \Big\}, \\
T(b) &= \Big\{ Y = X_a - X_b : a = 1, 2, \ldots, \hat{b}, \ldots, N \Big\}
\end{aligned} \tag{5.5}$$

for each $b = 1, 2, \ldots, N$. The notation $\hat{b}$ appearing in the definition $T(b)$ indicates that the $a = b$ term is omitted. As each of the double sums considered is absolutely convergent, it suffices to specify the set of points in $\mathbb{C}$ over which each sum is to be performed. In particular, we do not need to specify the order in which the component summations implicit in each term of (5.4) are to be taken. Note also that, in the second line of (5.4), we have used the fact that the $a = b$ term in the first series on the RHS vanishes because of (5.2).

As above we will consider a second lattice Γ' such that $\Gamma \subset \Gamma'$ with index N. By definition this means that there are N points of Γ' in each period parallelogram

[5] In this section we set all masses to unity.

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D_Γ of Γ: $|\Gamma' \cap D_\Gamma(z_0)| = N$ for all $z_0 \in \mathbb{C}$. Now we will consider a configuration such that

$$\Big\{X_a : a = 1, 2, \ldots, N\Big\} = \Gamma' \cap D_\Gamma(0)\,. \tag{5.6}$$

Taking into account the action of the Weyl group, this choice corresponds to a single point on the Coulomb branch.

The idea now is that, for this configuration, summing over a and over points in Γ is equivalent to summing over points in Γ'. To see this note that we have

$$\Gamma' = \Big\{Y = X_a - Z : Z \in \Gamma, a = 1, 2, \ldots, N\Big\}\,. \tag{5.7}$$

The lattice is invariant under translations by any lattice vector. Hence, as $X_b \in \Gamma'$, we also have

$$\Gamma' = \Big\{Y = X_a - X_b - Z : Z \in \Gamma, a = 1, 2, \ldots, N\Big\} \tag{5.8}$$

for each b. From the definitions of $S(b)$ and $T(b)$ we now see that

$$S(b) \cup T(b) = \Gamma' - \{0\}\,. \tag{5.9}$$

For each $b = 1, 2, \ldots, N$. Finally, using this relation and (5.2), we have

$$\frac{\partial \mathcal{W}}{\partial X_b} = \sum_{Y \in S_1} \frac{1}{Y^3} + \sum_{Y \in S_2} \frac{1}{Y^3} = \sum_{Y \in \Gamma' - \{0\}} \frac{1}{Y^3} = 0\,. \tag{5.10}$$

This shows that $\mathcal{W}$ has a critical point corresponding to each lattice Γ' which has Γ as a sublattice of index N. Hence the superpotential reproduces the set of vacua predicted by the Donagi-Witten curve. There are several loose ends here: we have not shown that the critical points found above are non-degenerate and thus correspond to massive vacua. Nor have we shown that they are unique. Finally, apart for the explicit calculation for gauge group SU(3) reported in section 3, we have not checked the existence of the massless vacua predicted by the classical analysis of section 2. These issues are under investigation.

In the next section we will need to know the explicit form for the vacuum configurations in the case where N is prime. In this case, by the analysis of section 1.1, we should find N vacua in the confining phase and a single vacuum in the Higgs phase. As above these each of these vacua should correspond to lattices Γ' such that $\Gamma \subset \Gamma'$ with index N. If Γ has periods $(2\omega_1, 2\omega_2)$ then two obvious solutions of this condition are the lattices Γ'_A with periods $(2\omega_1/N, 2\omega_2)$ and Γ'_B with periods $(2\omega_1, 2\omega_2/N)$. From section 1.1, the torus $E = \mathbb{C}/\Gamma$ has periods $2\omega_1 = 2\pi i$ and $2\omega_2 = 2\pi i\tau$. Thus, the two vacuum configurations corresponding to Γ'_A and Γ'_B are, $X_a^{(A)} = 2\pi i a/N$ and $X_a^{(B)} = 2\pi i\tau a/N$ respectively, with $a = 1, 2, \ldots, N$ in both cases. The coupling independence of configuration A identifies it as the vacuum of Higgs sector: this is the only vacuum which is visible in the semiclassical limit $\tau \to \infty$. The second vacuum must therefore be one of the N confining vacua. From the point of view of the four-dimensional $\mathcal{N} = 2$ theory, vacua A and B are associated with the condensation of elementary quanta and magnetic monopoles respectively. The remaining $N - 1$ confining vacua are generated by the modular transformation $\tau \to \tau + p$ for $p = 1, 2, \ldots, N$. In the $\mathcal{N} = 4$ theory, this transformation turns a magnetic monopole into a dyon which carries electric charge p. Correspondingly, for soft breaking of $\mathcal{N} = 2$ SUSY these vacua are associated with the condensation of massless BPS dyons.

6. An application

A simple application of the superpotential derived above is to calculate the condensates of gauge invariant chiral operators. An obvious candidate is the value of the modulus $u_2 = \langle \mathrm{Tr}\Phi^2 \rangle$ in each vacuum. According to the results of section 4, we have $u_2 = \mathcal{W}/\epsilon$, thus the vacuum values of u_2 are given by the critical values of the superpotential. There is a subtlety with this identification which is best illustrated in the SU(2) case. Recall that the three critical values of the superpotential were found in section 3 to be $e_1(\tau)$, $e_2(\tau)$ and $e_3(\tau)$. These three functions of τ transform under the modular group with weight two modulo permutations. However this result depended on setting the additive constant $C(\tau)$ appearing in (1.1) to zero. In particular we noted that any non-zero value for $C(\tau)$ would spoil the modular property of the superpotential. In fact this precisely matches a similar ambiguity in the parameter u discussed in [32]. If we retain the notation u for the classical modulus $\langle \mathrm{Tr}\Phi^2 \rangle$, the Seiberg-Witten curve can be parameterized by a coordinate $\tilde{u}$ which transforms with modular weight two. On general grounds the two coordinates are related as,[6]

$$\tilde{u} = u - M^2 \left(\frac{1}{12} + \sum_{p=1}^{\infty} \alpha_p q^p \right), \tag{6.1}$$

where the coefficients α_p are unknown for $p > 1$. If we choose $C(\tau) = 0$, the corresponding superpotential of modular weight two is therefore $\mathcal{W} = \epsilon \tilde{u}$. In view of (6.1), predictions for $\langle \mathrm{Tr}\Phi^2 \rangle$ in different vacua are only determined up to an overall additive function of τ. Thus, only predictions for differences between the values of u in distinct vacua are unamibguous. A similar additive ambiguity arises in the definition of u_2 for $N > 2$ but we will suppress it in the following. Note that the critical values of $\mathcal{W}$ also determine the gluino condensate via $\langle \mathrm{Tr}\lambda_\alpha \lambda^\alpha \rangle = \partial_\tau \mathcal{W}$. As the superpotential is independent of R, the results will apply equally in the four-dimensional limit.

[6] I would like to thank Matt Strassler for pointing out the incorrect discussion of this point given in a previous version of this paper.

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In the following we will restrict our attention to the massive vacua and then to the simplest case of prime N. To compute the critical value of the superpotential in the Higgs vacuum given in the previous section, we must compute the sum

$$u_2^{(A)}/M^2 = \mathcal{W}|_{X=X^{(A)}} = \frac{1}{2}\sum_{b=1}^{N}\sum_{a\neq b}^{N}\mathcal{P}\left(\frac{2(a-b)}{N}i\pi\right) = \frac{N}{2}\sum_{d=1}^{N-1}\mathcal{P}\left(\frac{2\pi i d}{N}\right). \tag{6.2}$$

As we have $\omega_1 = i\pi$ and $\omega_2 = i\pi\tau$, the Weierstrass function can be written as

$$\mathcal{P}(2\pi i z) = -\frac{1}{4\pi^2}\left(\frac{1}{z^2} + \sum_{(m.n)\neq(0,0)}\left[\frac{1}{(z-m-n\tau)^2} - \frac{1}{(m+n\tau)^2}\right]\right). \tag{6.3}$$

The double series on the RHS is absolutely convergent and therefore independent of the order of summation over m and n. However, some care is required because, when taken in isolation, either of the two terms in the summand leads to a double series which is only conditionally convergent and, in particular, depends the order of summation. To illustrate this we consider the second term which, for one choice of ordering, is proportional the regulated second Eisenstein series [22]

$$E_2(\tau) = \frac{3}{\pi^2}\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}{}' \frac{1}{(m+n\tau)^2}\,. \tag{6.4}$$

The prime on the m summation means 'omit the $m=0$ term when $n=0$'. If we naively interchange the order of summation and multiply both sides by τ^2 we would obtain the relation $E_2(-1/\tau) = \tau^2 E_2(\tau)$. As E_2 is also invariant under $\tau \to \tau+1$, this would lead us to conclude incorrectly that E_2 is a modular form of weight two. In fact the summations over m and n fail to commute and instead give an anomalous transformation law for E_2 under the modular transformation $\tau \to -1/\tau$

$$E_2(\tau) = \frac{1}{\tau^2}E_2\left(-\frac{1}{\tau}\right) - \frac{6}{\pi i \tau}\,. \tag{6.5}$$

Paying attention to the order of summation we obtain

$$\begin{aligned}
&u_2^{(A)}/M^2 = \\
&= -\frac{N}{8\pi^2}\left[\sum_{d=1}^{N-1}\frac{N^2}{d^2} + \sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}{}'\left(\left[\sum_{d=1}^{N}\frac{N^2}{(mN-d+nN\tau)^2}\right] - \frac{N}{(m+n\tau)^2}\right)\right] = \\
&= -\frac{N}{8\pi^2}\left[\sum_{l=-\infty}^{\hat{+\infty}}\frac{N^2}{l^2} + \sum_{n=-\infty}^{\hat{+\infty}}\sum_{l=-\infty}^{+\infty}\frac{N^2}{(l+nN\tau)^2} - \frac{N\pi^2}{3}E_2(\tau)\right] = \\
&= \frac{N^2}{24}\Big(E_2(\tau) - NE_2(N\tau)\Big).
\end{aligned} \tag{6.6}$$

where the term where the summation variable equals zero is omitted in each of the hatted sums. A similar calculation yields the expectation value of u_2 in the confining vacuum corresponding to the lattice Γ'_B as

$$u_2^{(B)}/M^2 = \mathcal{W}|_{X=X^{(B)}} = \frac{N^2}{24}\left(E_2(\tau) - \frac{1}{N}E_2\left(\frac{\tau}{N}\right)\right). \tag{6.7}$$

The value of u_2 in the remaining $N-1$ confining vacua is obtained by replacing τ with $\tau+p$ in (6.7) for $p = 1, 2, \ldots, N-1$. Using eq. (6.5) we find that $u_2^{(A)}(-1/\tau) = \tau^2 u_2^{(B)}(\tau)$. Notice that the anomalous term in the transformation law for E_2 cancels between the two terms contributing to $u_2^{(A)}$. Hence we find that the vacuum values of u_2 tranform with weight two under, and are also interchanged by, an electric-magnetic duality transformation. This generalizes the SU(2) result for$\tilde{u}$ described above. The gluino condensate $S = \langle \mathrm{Tr}\lambda_\alpha\lambda^\alpha\rangle$ in each vacuum is obtained by a differentiating the corresponding expression for u_2 with respect to τ. As this differentiation does not commute with a general $SL(2,\mathbb{Z})$ tranformation, the resulting functions of τ do not have any special modular properties. Finally, we note that this calculation can easily be generalized to the condensates u_n with $n > 2$ [34].

7. Conclusion

The main result of this paper is that the exact superpotential of the $\mathcal{N}=1$ theory obtained by introducing chiral multiplet masses in $\mathcal{N}=4$ supersymmetric SU(N) Yang-Mills theory on $\mathbb{R}^3 \times S^1$ is

$$\mathcal{W} = m_1 m_2 m_3 \sum_{a>b} \mathcal{P}(X_a - X_b)\,. \tag{7.1}$$

An interesting feature of this result is that the superpotential coincides with the complexified potential of the elliptic Calogero-Moser system. The same integrable system is associated with $\mathcal{N}=2$ supersymmetric Yang-Mills theory with one adjoint hypermultiplet of mass M. Specifically the spectral curve which encodes the conserved quantities of the Calogero-Moser system is the same complex curve which governs the Coulomb branch of the four-dimensional $\mathcal{N}=2$ theory. This suggests a new connection between $\mathcal{N}=1$ theories and integrable systems which extends the correspondence between $\mathcal{N}=2$ theories and integrable systems introduced in [3, 27]. Starting from an $\mathcal{N}=2$ theory in four dimensions, the superpotential of the $\mathcal{N}=1$ theory on $\mathbb{R}^3 \times S^1$ obtained by introducing a mass for the adjoint scalar in the $\mathcal{N}=2$ vector multiplet shouldcoincide with the complexified potential of the corresponding integrable system. This connection should be a general one which applies to all $\mathcal{N}=1$ theories obtained by soft breaking of $\mathcal{N}=2$ supersymmetry. Following the arguments given in section 4, the correspondence can be 'explained' by combining two previous observations:

(1) The Coulomb branch of an $\mathcal{N} = 2$ theory on $\mathbb{R}^3 \times S^1$ coincides with the complexified phase space of the integrable hamiltonian system associated with the corresponding theory on $\mathbb{R}^4$ [28]. The dynamical variables of the integrable system, $\{X^a, P^a\}$ yield holomorphic coordinates on the Coulomb branch of the compactified theory.

(2) The complex moduli u_n, $n = 2, \ldots, N$, which parametrize the four dimensional Coulomb branch correspond to the conserved quantities of the integrable system [3]. In particular, we may identify u_2 with the hamiltonian, $\mathcal{H}(X^a, P^a)$, of the integrable system (see footnote 3 above). Introducing a superspace mass term $\mathcal{W} = \epsilon u_2$ for the adjoint scalar in the $\mathcal{N} = 2$ vector multiplet, we obtain an $\mathcal{N} = 1$ theory on $\mathbb{R}^3 \times S^1$ with superpotential $\mathcal{W} = \epsilon\mathcal{H}(X^a, P^a)$. As the momenta appear via the non-relativistic kinetic terms,[7] $\sum P_a^2/2$, they can be trivially eliminated via their equations of motion and the resulting superpotential is just the complexified potential of the integrable system.

One case where we can immediately make contact with known results is for the $\mathcal{N} = 2$ theory without adjoint hypermultiplets. For gauge group SU(N), the relevant integrable system is the affine Toda lattice [27]. The exact superpotential of the corresponding $\mathcal{N} = 1$ theory on $\mathbb{R}^3 \times S^1$ is known to coincide with the affine Toda potential [36]. In fact, this superpotential can also be obtained directly from (7.1) by decoupling the three adjoint chiral multiplets. The appropriate limit, which is well known on the integrable system side of the correspondence [37], generalizes the limit discussed in section 3 for the SU(2) case. It would be interesting to test this correspondence for other $\mathcal{N} = 2$ theories for which the corresponding integrable system is known. The correspondence may also suggest new results in cases where the relevant integrable system is unknown.

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[7] This point would need to be modified if the correspondence is to apply also to $\mathcal{N} = 2$ theories obtained by compactification on a circle from five dimensions [35], where the corresponding integrable systems are relativistic.

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